MATH5835M Statistical Computing
Exercise Sheet 5 (answers)

https://www1.maths.leeds.ac.uk/~voss/2023/MATH5835M/
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Answer 12. In a transition matrix, the rows must sum to 1 and therefore we need $\alpha_1 = 0.6$, $\alpha_2 = 0.1$ and $\alpha_3 = 0.6$. From lectures we know that the condition for π to be a stationary distribution is $\pi^\top P = \pi^\top$, *i.e.*

$$(\pi_1\ \pi_2\ \pi_3) \begin{pmatrix} 0.4 & 0.6 & 0.0 \\ 0.3 & 0.1 & 0.6 \\ 0.0 & 0.6 & 0.4 \end{pmatrix} = (\pi_1\ \pi_2\ \pi_3). \tag{1}$$

Together with the condition that π is a probability vector, we get a system of four equations:

$$\begin{aligned} \frac{4}{10}\pi_1 + \frac{3}{10}\pi_2 &= \pi_1, \\ \frac{6}{10}\pi_1 + \frac{1}{10}\pi_2 + \frac{6}{10}\pi_3 &= \pi_2, \\ \frac{6}{10}\pi_2 + \frac{4}{10}\pi_3 &= \pi_3, \\ \pi_1 + \pi_2 + \pi_3 &= 1. \end{aligned}$$

We have four equations for three unknowns, so one of the equations is redundant. Leaving out any of the first three equations, we can solve this system to get

$$\pi_1 = 1/5, \quad \pi_2 = 2/5, \quad \pi_3 = 2/5.$$

An alternative way to obtain the same solution is to observe the fact that equation (1) implies that π is an eigenvector of $P^\top$ with eigenvalue 1. Using R we get the same result as above:

```
> P <- matrix(c(.4, .3, 0, .6, .1, .6, 0, .6, .4), 3, 3)
> P
     [,1] [,2] [,3]
[1,]  0.4  0.6  0.0
[2,]  0.3  0.1  0.6
[3,]  0.0  0.6  0.4
> eigen(t(P))
eigen() decomposition
$values
[1]  1.0 -0.5  0.4

$vectors
          [,1]       [,2]          [,3]
[1,] 0.3333333  0.2672612  7.071068e-01
[2,] 0.6666667 -0.8017837 -3.561232e-16
[3,] 0.6666667  0.5345225 -7.071068e-01

> pi <- eigen(t(P))$vectors[,1]
> pi / sum(pi)
[1] 0.2 0.4 0.4
```

Answer 13. Rather than working with the definition of a stationary density directly, for an AR(1) process it is easier to use the fact that all X_k are normally distributed, and to just find the mean and variance which make the process stationary. From this we can then get the required density.

Assume that $X_{k-1} \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$\mathbb{E}(X_k) = \mathbb{E}\bigl(\alpha X_{k-1} + \varepsilon_k\bigr) = \alpha \mathbb{E}(X_{k-1}) + \mathbb{E}(\varepsilon_k) = \alpha\mu + 0 = \alpha\mu$$

and

$$\operatorname{Var}(X_k) = \operatorname{Var}\bigl(\alpha X_{k-1} + \varepsilon_k\bigr) = \alpha^2 \operatorname{Var}(X_{k-1}) + \operatorname{Var}(\varepsilon_k) = \alpha^2\sigma^2 + 1.$$

If the process is stationary, X_k has the same distribution as X_{k-1}, and in particular has the same mean and variance. From this we get the two equations $\mu = \alpha\mu$ and $\sigma^2 = \alpha^2\sigma^2 + 1$. Solving these equations, we find $\mu = 0$ and $\sigma^2 = 1/(1-\alpha^2)$. Thus the stationary distribution of the process is $\mathcal{N}\bigl(0, 1/(1-\alpha^2)\bigr)$, with density

$$\pi(x) = \sqrt{\frac{1-\alpha^2}{2\pi}} \exp\Bigl(-\frac{1}{2}(1-\alpha^2)x^2\Bigr).$$

Answer 14. The initial distribution is the distribution of X_0, and thus is the standard normal distribution $\mathcal{N}(0,1)$, with density

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\bigl(-x^2/2\bigr)$$

for all $x \in \mathbb{R}$. The transition density is the the density of X_{n+1}, given $X_n = x$. Since $X_{n+1} \sim \mathcal{N}(0,1)$, irrespective of the value of X_n, we find the transition density as

$$p(x,y) = \varphi(y)$$

for all $x, y \in \mathbb{R}$.

Answer 15. The general formula for the acceptance probability is

$$\alpha(x,y) = \min\Bigl(1, \frac{\pi(y)p(y,x)}{\pi(x)p(x,y)}\Bigr).$$

The target density π is given to be

$$\pi(x) = \frac{1}{Z} \sin(x)^2 \exp(-|x|)$$

and we are considering three different transition densities p.

a) Here we have $p(x,y) = \frac{1}{\sqrt{2\pi}} \exp\bigl(-(y-x)^2/2\bigr)$. In this case, we have $p(x,y) = p(y,x)$ and thus

$$\alpha(x,y) = \min\Bigl(1, \frac{\pi(y)}{\pi(x)}\Bigr) = \min\Bigl(1, \frac{\sin(y)^2 \exp(-|y|)}{\sin(x)^2 \exp(-|x|)}\Bigr).$$

This is an example of the Random Walk Metropolis algorithm.

b) Here we have $p(x,y) = 1_{[x-1,x+2]}(y)/3$ and thus

$$\alpha(x,y) = \min\Bigl(1, \frac{\sin(y)^2 \exp(-|y|) 1_{[y-1,y+2]}(x)}{\sin(x)^2 \exp(-|x|) 1_{[x-1,x+2]}(y)}\Bigr).$$

Since y is always taken to be a proposal, sampled with density $p(x,\cdot)$, we always have $y \in [x-1, x+2]$ and we can simplify the denominator to

$$\alpha(x,y) = \min\Bigl(1, \frac{\sin(y)^2 \exp(-|y|) 1_{[y-1,y+2]}(x)}{\sin(x)^2 \exp(-|x|)}\Bigr).$$

c) Here we have $p(y) = \frac{1}{\sqrt{2\pi}} \exp\bigl(-y^2/2\bigr)$ and thus

$$\alpha(x,y) = \min\Bigl(1, \frac{\sin(y)^2 \exp(-|y|) \exp(-x^2/2)}{\sin(x)^2 \exp(-|x|) \exp(-y^2/2)}\Bigr).$$

This is an example of the independence sampler.