MATH5835M Statistical Computing
Exercise Sheet 4 (answers)

https://www1.maths.leeds.ac.uk/~voss/2023/MATH5835M/
Jochen Voss, J.Voss@leeds.ac.uk
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Answer 9.

a) From lectures we know $\mathbb{E}\big(\tilde Z_N\big) = \mathbb{E}\big(f(X)\big)$, where $X \sim \mathcal{N}(0, \sigma^2)$. Writing this as an integral, we find

$$\mathbb{E}\big(\tilde Z_N\big) = \int_{-\infty}^{\infty} f(x)\, \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-x^2/2\sigma^2)\, dx.$$

b) To cancel the density of the normal distribution in the preceding integral, we can choose

$$Z_N = \frac{1}{N} \sum_{j=1}^{N} \sqrt{2\pi\sigma^2} \exp(X_j^2/2\sigma^2) f(X_j),$$

i.e. $g(x) = \sqrt{2\pi\sigma^2}\exp(x^2/2\sigma^2) f(x)$. This estimator has $\mathbb{E}\big(Z_N\big) = \int_{-\infty}^{\infty} f(x)\, dx$ as required.

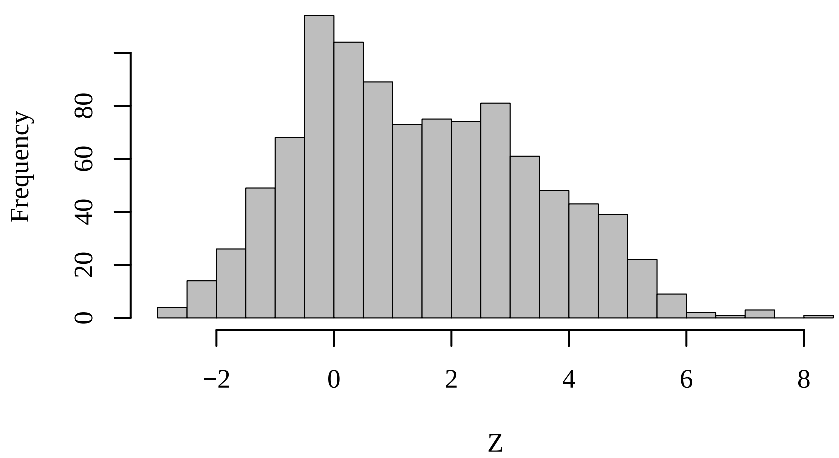
c) Since Z_N is an unbiased estimator, we have

$$\begin{aligned}
\mathrm{MSE}(Z_N) &= \mathrm{Var}(Z_n) \\
&= \frac{1}{N} \mathrm{Var}\Big(\sqrt{2\pi\sigma^2}\exp(X^2/2\sigma^2) f(X)\Big) \\
&= \frac{2\pi\sigma^2}{N} \mathbb{E}\Big(\exp(X^2/\sigma^2) f(X)^2\Big) - \frac{1}{N}\Big(\int_{-\infty}^{\infty} f(x)\, dx\Big)^2 \\
&= \frac{\sqrt{2\pi\sigma^2}}{N} \int_{-\infty}^{\infty} f(x)^2 \exp(x^2/2\sigma^2)\, dx - \frac{1}{N}\Big(\int_{-\infty}^{\infty} f(x)\, dx\Big)^2.
\end{aligned}$$

Answer 10. We can use $U \sim \mathcal{U}[0,1]$ together with the `ifelse()` function to decide whether $Z = X$ or $Z = Y$.

```
N <- 1000
X <- rnorm(N, 0, 1)
Y <- rnorm(N, 3, sqrt(2))
U <- runif(N)
Z <- ifelse(U < 1/2, X, Y)
hist(Z, breaks=20, main=NULL, col="grey")
```

This gives a histogram like the following:



Answer 11.

a) Proposals have density

$$g(x) = \exp(-x)$$

for all $x \geq 0$. The acceptance probability is

$$p(x) = \sin(x)^2.$$

From the basic rejection sampling lemma we know that accepted samples then have density

$$f(x) = \frac{1}{Z} p(x) g(x) = \frac{1}{Z} \sin(x)^2 \exp(-x),$$

where Z is the normalizing constant.

b) Since we are given the acceptance probability, we can use basic rejection sampling. Since we don't know in advance how many proposals we will need, we use a `while` loop to create the samples:

```
N <- 1000
X <- NULL # empty vector
while (length(X) < N) {
    Xi <- rexp(1)
    Ui <- runif(1)
    if (Ui <= sin(Xi)^2) {
        X <- c(X, Xi)
    }
}
hist(X, breaks=20, main=NULL, col="grey")
```

The resulting histogram looks as follows: